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EVOLUTIONARY EQUATION FOR PERTURBATIONS IN A TWO-LAYER FILM FLOW

O. Yu. Tsvelodub

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We will examine the simultaneous motion of two films of immiscible liquids flowing under the influence of gravity. Such flows are encountered in certain types of extraction columns. The chosen coordinate system is shown in Fig. 1. The film bounded by the solid wall will henceforth be designated as the first film, while the film having the free boundary will be referred to as the second film. The quantities pertaining to these films will be denoted by the subscripts 1 and 2, respectively.

The equations which describe the motion of such a system permit a solution to be obtained with plane phase and free boundaries, regardless of the rates of flow of the liquids. Here, the profiles of longitudinal velocity are equal to

$$U_{10} = \frac{g}{2\nu_1}[2(H_{10} + H_{20}\rho_2/\rho_1)y - y^2],$$
$$U_{20} = \frac{g}{2\nu_2}[2H_{10}H_{20}(\mu_2/\mu_1 - 1) + H_{10}^2(\nu_2/\nu_1 - 1) + 2(H_{10}+H_{20})y - y^2]. \quad (1)$$

Here, ν_i and μ_i are the kinematic and absolute viscosities; ρ_i is density; H_{i0} is the thickness of the liquid film.

However, even with low flow rates, the flow (1) may become wavelike due to instability. Using as scales characteristic values of the quantities pertaining to the first film – especially the thickness H_{10} and the mean-flow-rate velocity U_0 – for nonwavy flow with the

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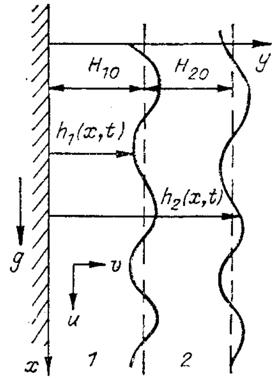

Fig. 1

given flow rate q_0, we write the equations of motion in dimensionless form for this case (with omission of the symbol denoting that the quantities have been made dimensionless)

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = \left(-\frac{\partial p_1}{\partial x} + \frac{\mathrm{g}}{g}\right)\Big/\mathrm{Fr} + \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right)\Big/\mathrm{Re},\\
&\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y}\Big/\mathrm{Fr} + \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right)\Big/\mathrm{Re},\\
&\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0,\\
&\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} = \left(-\frac{\partial p_2}{\partial x} + \frac{\mathrm{g}}{g}\right)\Big/\rho\mathrm{Fr} + \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right)\nu/\mathrm{Re},\\
&\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} = -\frac{\partial p_2}{\partial y}\Big/\rho\mathrm{Fr} + \left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}\right)\nu/\mathrm{Re},\\
&\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0.
\end{aligned} \tag{2}$$

The dynamic boundary conditions on the solid boundary ($y = 0$), phase boundary ($y = h_1(x, t)$), and free boundary ($y = h_2(x, t)$) can be written in a form similar to [1]:

$$\begin{gathered}
\text{at } y = 0 \quad u_1 = v_1 = 0;\\
\text{at } y = h_1(x, t) \quad u_1 = u_2,\ v_1 = v_2,\\
-(p_1 - \mathrm{We}/R_1)\frac{\partial h_1}{\partial x} + \left(2\frac{\partial u_1}{\partial x}\frac{\partial h_1}{\partial x} - \frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x}\right)\mathrm{Fr/Re} = -p_2\frac{\partial h_1}{\partial x} + \left(2\frac{\partial u_2}{\partial x}\frac{\partial h_1}{\partial x} - \frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x}\right)\mu\mathrm{Fr/Re},\\
p_1 - \mathrm{We}/R_1 + \left[\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\frac{\partial h_1}{\partial x} - 2\frac{\partial v_1}{\partial y}\right]\mathrm{Fr/Re} = p_2 + \left[\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right)\frac{\partial h_1}{\partial x} - 2\frac{\partial v_2}{\partial y}\right]\mu\mathrm{Fr/Re};
\end{gathered} \tag{3}$$

at $y = h_2(x, t)$

$$\begin{gathered}
-(p_2 - \sigma\mathrm{We}/R_2)\frac{\partial h_2}{\partial x} + \left(2\frac{\partial u_2}{\partial x}\frac{\partial h_2}{\partial x} - \frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x}\right)\mu\mathrm{Fr/Re} = -p_0\frac{\partial h_2}{\partial x},\\
p_2 - \sigma\mathrm{We}/R_2 + \left[\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right)\frac{\partial h_2}{\partial x} - 2\frac{\partial v_2}{\partial y}\right]\mu\mathrm{Fr/Re} = p_0,
\end{gathered}$$

where

$$\frac{1}{R_1} = -\frac{\partial^2 h_1}{\partial x^2}\Big/\left(1 + \left(\frac{\partial h_1}{\partial x}\right)^2\right)^{3/2}; \quad \frac{1}{R_2} = -\frac{\partial^2 h_2}{\partial x^2}\Big/\left(1 + \left(\frac{\partial h_2}{\partial x}\right)^2\right)^{3/2};$$
$$\mathrm{Re} = q_0/\nu_1; \quad \mathrm{Fr} = q_0^2/gH_{10}^3; \quad \mathrm{We} = \sigma_1/\rho_1 g H_{10}^2.$$

The below kinematic conditions are valid at the phase and free boundaries

$$\begin{aligned}
&\partial h_1/\partial t + u_1\partial h_1/\partial x = v_1 \quad \text{at } y = h_1;\\
&\partial h_2/\partial t + u_2\partial h_2/\partial x = v_2 \quad \text{at } y = h_2.
\end{aligned} \tag{4}$$

Here, Re is the Reynolds number; Fr is the Froude number; We is the Weber number; $\rho = \rho_2/\rho_1$, $\sigma = \sigma_2/\sigma_1$, $\nu = \nu_2/\nu_1$, $\mu = \mu_2/\mu_1$ are relative values of density, surface tension, and kinematic and absolute viscosity.

Using (1), it is not hard to show that with the chosen scales for making quantities dimensionless, the following relation is satisfied

$$\mathrm{Fr} / \mathrm{Re} = 1/3 + \rho h/2 \equiv 1/a.$$

Limiting ourselves to the examination of longwave disturbances, we seek the solution of system (2) with conditions (3-4) in the form of series in the small parameter $\varepsilon = H_{10}/\lambda$ (where λ represents the characteristic and longitudinal dimension of the disturbances). For this, we follow [1] and introduce the new variables

$$x' = \varepsilon x, \quad y' = y, \; \tau_n = \varepsilon^n t, \; n = 1, 2\ldots$$

and functions

$$u_1 = U_{10} + \varepsilon u_1', \; v_1 = \varepsilon^2 v'_1, \; u_2 = U_{20} + \varepsilon u_2', \; v_2 = \varepsilon^2 v'_2,$$
$$p_1 = \varepsilon p_1', \; p_2 = \varepsilon p_2', \; h_1 = 1 + \varepsilon h_1', \; h_2 = h + \varepsilon h_2'$$
$$(h = 1 + H_{20}/H_{10}).$$

Ignoring terms on the order of ε^2 and above and changing the boundary conditions for the phase and free boundaries to their undisturbed levels, for u_1', v_1', u_2', v_2', p_1', p_2', h_1', h_2' we arrive at the system (with the primes omitted)

$$\begin{gathered}
\varepsilon\left(\frac{\partial u_1}{\partial \tau_1} + U_{10}\frac{\partial u_1}{\partial x} + v_1\frac{dU_{10}}{dy}\right) = -\frac{\varepsilon}{\mathrm{Fr}}\frac{\partial p_1}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u_1}{\partial y^2},\\
-\frac{1}{\mathrm{Fr}}\frac{\partial p_1}{\partial y} + \frac{\varepsilon}{\mathrm{Re}}\frac{\partial^2 v_1}{\partial y^2} = 0, \quad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0,\\
\varepsilon\left(\frac{\partial u_2}{\partial \tau_1} + U_{20}\frac{\partial u_2}{\partial x} + v_2\frac{dU_{20}}{dy}\right) = -\frac{\varepsilon}{\rho\mathrm{Fr}}\frac{\partial p_2}{\partial x} + \frac{\nu}{\mathrm{Re}}\frac{\partial^2 u_2}{\partial y^2},\\
-\frac{1}{\rho\mathrm{Fr}}\frac{\partial p_2}{\partial y} + \frac{\varepsilon\nu}{\mathrm{Re}}\frac{\partial^2 v_2}{\partial y^2} = 0,\\
\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0
\end{gathered} \tag{5}$$

with the following boundary conditions:

$$\begin{gathered}
\text{at } y = 0 \quad u_1 = v_1 = 0;\\
\text{at } y = 1\\
\left(\varepsilon\frac{\partial u_1}{\partial y} + \frac{dU_{10}}{dy}\right)h_1 + u_1 = \left(\varepsilon\frac{\partial u_2}{\partial y} + \frac{dU_{20}}{dy}\right)h_1 + u_2, \quad v_1 = v_2,\\
\left(\frac{d^2U_{10}}{dy^2} + \varepsilon\frac{\partial^2 u_1}{\partial y^2}\right)h_1 + \frac{\partial u_1}{\partial y} = \mu\left[\left(\frac{d^2U_{20}}{dy^2} + \varepsilon\frac{\partial^2 u_2}{\partial y^2}\right)h_1 + \frac{\partial u_2}{\partial y}\right],\\
p_1 + \varepsilon\frac{\partial p_1}{\partial y}h_1 + \mathrm{We}\varepsilon^2\frac{\partial^2 h_1}{\partial x^2} - 2\varepsilon\frac{\mathrm{Fr}}{\mathrm{Re}}\frac{\partial v_1}{\partial y} = p_2 - 2\varepsilon\frac{\mu\mathrm{Fr}}{\mathrm{Re}}\frac{\partial v_2}{\partial y};\\
\text{at } y = h\\
\frac{d^2U_{20}}{dy^2}h_2 + \frac{\partial u_2}{\partial y} + \varepsilon\frac{\partial^2 u_2}{\partial y^2}h_2 = 0,\\
p_2 + \varepsilon\frac{\partial p_2}{\partial y}h_2 + \mathrm{We}\sigma\varepsilon^2\frac{\partial^2 h_2}{\partial x^2} - 2\varepsilon\frac{\mu\mathrm{Fr}}{\mathrm{Re}}\frac{\partial v_2}{\partial y} = p_0.
\end{gathered} \tag{6}$$

Kinematic conditions (4) take the form

$$\begin{aligned}
&\frac{\partial h_1}{\partial \tau_1} + \varepsilon\frac{\partial h_1}{\partial \tau_2} + U_{10}\frac{\partial h_1}{\partial x} + \frac{dU_{10}}{dy}\varepsilon h_1\frac{\partial h_1}{\partial x} + \varepsilon u_1\frac{\partial h_1}{\partial x} = v_1 + \varepsilon\frac{\partial v_1}{\partial y}h_1 \quad \text{at } y = 1,\\
&\frac{\partial h_2}{\partial \tau_1} + \varepsilon\frac{\partial h_2}{\partial \tau_2} + U_{20}\frac{\partial h_2}{\partial x} + \varepsilon u_2\frac{\partial h_2}{\partial x} = v_2 + \varepsilon\frac{\partial v_2}{\partial y}h_2 \quad \text{at } y = h.
\end{aligned} \tag{7}$$

The terms of higher order with respect to ε are left in (6), since the values of We are usually large for the thin films of many liquids. We will therefore suppose that the following relations are satisfied

$$\mathrm{We}\,\varepsilon^2 \sim 1,\ \mathrm{We}\varepsilon^2\sigma \sim 1.$$

Representing the solution of system (5) in the form of a series in

$$\begin{matrix} u_i \\ v_i \\ p_i \\ h_i \end{matrix} = \sum_{k=0}^{\infty} \varepsilon^k \begin{pmatrix} u_{ik} \\ v_{ik} \\ p_{ik} \\ h_{ik} \end{pmatrix}, \quad i = 1, 2,$$

and equating the coefficients with identical powers of ε, we obtain the following from the equations for the zeroth order

$$u_{10} = r_1 y,\ \ v_{10} = -\frac{\partial r_1}{\partial x}\frac{y^2}{2},\ \ u_{20} = r_2 y + r_3,\ \ v_{20} = -\frac{\partial r_2}{\partial x}\frac{y^2}{2} - \frac{\partial r_3}{\partial x} y - r_4. \qquad (8)$$

The explicit form of the functions r_i is given in the appendix.

Inserting (8) into (7), we arrive at a system which in a first approximation describes the behavior of disturbances in a two-layer film flow:

$$\frac{\partial h_{10}}{\partial \tau_1} + a_1 \frac{\partial h_{10}}{\partial x} + b_1 \frac{\partial h_{20}}{\partial x} = 0,\ \ \frac{\partial h_{20}}{\partial \tau_1} + a_2 \frac{\partial h_{20}}{\partial x} + b_2 \frac{\partial h_{10}}{\partial x} = 0. \qquad (9)$$

Here

$$a_1 = U_{10}(1) + \frac{1}{2}\left(\mu \frac{d^2U_{20}(1)}{dy^2} - \frac{d^2U_{10}(1)}{dy^2}\right);\quad b_1 = -\frac{\mu}{2}\frac{d^2U_{20}(h)}{dy^2};$$

$$a_2 = U_{20}(h) + \frac{d^2U_{20}(h)}{dy^2}\left[(\mu - 1)/2 + (1-\mu)h - h^2/2\right];$$

$$b_2 = \left(\frac{dU_{10}(1)}{dy} - \frac{dU_{20}(1)}{dy}\right)(h-1) + \left(\mu \frac{d^2U_{20}(1)}{dy^2} - \frac{d^2U_{10}(1)}{dy^2}\right)\left(h - \frac{1}{2}\right).$$

The general solution of system (9) is easily calculated:

$$h_{10} = H_{11}(\xi_1) + H_{12}(\xi_2),\ h_{20} = H_{21}(\xi_1) + H_{22}(\xi_2),\ \xi_1 = x - c_1\tau_1,\ \xi_2 = x - c_2\tau_1$$
$$(c_{1,2} = [a_1 + a_2 \pm ((a_1 - a_2)^2 + 4b_1b_2)^{1/2}]/2).$$

It is clear that (9) has solutions in the form of steady traveling waves with the phase velocity c_1 or c_2.

We obtain the following from (5) for the next order with respect to ε

$$\begin{gathered}
\frac{\partial u_{10}}{\partial \tau_1} + U_{10}\frac{\partial u_{10}}{\partial x} + v_{10}\frac{dU_{10}}{dy} = -\frac{1}{\mathrm{Fr}}\frac{\partial p_{10}}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u_{11}}{\partial y^2},\\
-\frac{1}{\mathrm{Fr}}\frac{\partial p_{11}}{\partial y} + \frac{1}{\mathrm{Re}}\frac{\partial^2 v_{10}}{\partial y^2} = 0,\ \ \frac{\partial u_{11}}{\partial x} + \frac{\partial v_{11}}{\partial y} = 0,\\
\frac{\partial u_{20}}{\partial \tau_1} + U_{20}\frac{\partial u_{20}}{\partial x} + v_{20}\frac{dU_{20}}{dy} = -\frac{1}{\rho \mathrm{Fr}}\frac{\partial p_{20}}{\partial x} + \frac{\nu}{\mathrm{Re}}\frac{\partial^2 u_{21}}{\partial y^2},\\
-\frac{1}{\rho \mathrm{Fr}}\frac{\partial p_{21}}{\partial y} + \frac{\nu}{\mathrm{Re}}\frac{\partial^2 v_{20}}{\partial y^2} = 0,\ \ \frac{\partial u_{21}}{\partial x} + \frac{\partial v_{21}}{\partial y} = 0.
\end{gathered} \qquad (10)$$

It is easy to find a solution of system (10) that satisfies the boundary conditions which follow from (6)
at $y = 0$ $u_{11} = v_{11} = 0$;
at $y = 1$

$$\frac{\partial u_{10}}{\partial y} h_{10} + \frac{dU_{10}}{dy} h_{11} + u_{11} = \frac{\partial u_{20}}{\partial y} h_{10} + \frac{dU_{20}}{dy} h_{11} + u_{21},$$

$$v_{11} = v_{21}, \quad \frac{d^2U_{10}}{dy^2} h_{11} + \frac{\partial u_{11}}{\partial y} = \mu \left(\frac{d^2U_{20}}{dy^2} h_{11} + \frac{\partial u_{21}}{\partial y} \right),$$

$$p_{11} + \frac{\partial p_{10}}{\partial y} h_{10} + \mathrm{We}\varepsilon^2 \frac{\partial^2 h_{11}}{\partial x^2} - \frac{2\mathrm{Fr}}{\mathrm{Re}} \frac{\partial v_{10}}{\partial y} = p_{21} - \frac{2\mu\mathrm{Fr}}{\mathrm{Re}} \frac{\partial v_{20}}{\partial y};$$

at y = h

$$\frac{d^2U_{20}}{dy^2} h_{21} + \frac{\partial u_{21}}{\partial y} = 0, \quad p_{21} + \frac{\partial p_{20}}{\partial y} h_{20} + \mathrm{We}\sigma\varepsilon^2 \frac{\partial^2 h_{21}}{\partial x^2} - 2 \frac{\mu\mathrm{Fr}}{\mathrm{Re}} \frac{\partial v_{20}}{\partial y} = 0,$$

However, this solution is not presented here due to its awkwardness. If we insert it into the below relations, obtained from kinematic conditions for the given order with respect to

$$\frac{\partial h_{10}}{\partial \tau_2} + \frac{\partial h_{11}}{\partial \tau_1} + U_{10} \frac{\partial h_{11}}{\partial x} + \frac{dU_{10}}{dy} h_{10} \frac{\partial h_{10}}{\partial x} + u_{10} \frac{\partial h_{10}}{\partial x} = v_{11} + \frac{\partial v_{10}}{\partial y} h_{10} \quad \text{at} \quad y = 1,$$

$$\frac{\partial h_{20}}{\partial \tau_2} + \frac{\partial h_{21}}{\partial \tau_1} + U_{20} \frac{\partial h_{21}}{\partial x} + u_{20} \frac{\partial h_{20}}{\partial x} = v_{21} + \frac{\partial v_{20}}{\partial y} h_{20} \quad \text{at} \quad y = h,$$

after some simple but lengthy calculations we arrive at the system

$$\begin{aligned} &\frac{\partial h_{11}}{\partial \tau_1} + a_1 \frac{\partial h_{11}}{\partial x} + b_1 \frac{\partial h_{21}}{\partial x} = f_1(h_{10}, h_{20}), \\ &\frac{\partial h_{21}}{\partial \tau_1} + a_2 \frac{\partial h_{21}}{\partial x} + b_2 \frac{\partial h_{11}}{\partial x} = f_2(h_{10}, h_{20}) \end{aligned} \tag{11}$$

(f_1 and f_2 are functions of h_{10} and h_{20}, and their explicit form is given in the appendix).

For the solution of inhomogeneous system (11) to exist, it is necessary that the functions f_1 and f_2 satisfy certain solvability conditions [2]. These conditions in turn depend on the solutions of (11) and the corresponding homogeneous system (9) that are being examined. We will henceforth be interested only in steady traveling waves in a first approximation (i.e., dependent on the variables x and τ_1 in combination with ξ_1 or ξ_2), periodic waves, or isolated waves. In this case, the condition of solvability of system (11) requires that f_1 and f_2 satisfy the relation

$$b_2 f_1 - (a_1 - c_i) f_2 = 0. \tag{12}$$

Excluding H_{2i} from (12), we can use a relation which follows from (9) for a steady traveling wave

$$H_{2i} = \frac{c_i - a_1}{b_1} H_{1i} \tag{13}$$

to arrive at a single equation for H_{1i}:

$$\frac{\partial H_{1i}}{\partial \tau_2} + A H_{1i} \frac{\partial H_{1i}}{\partial \xi_i} + \mathrm{Re}\, B \frac{\partial^2 H_{1i}}{\partial \xi_i^2} + \mathrm{We}\, N \frac{\partial^4 H_{1i}}{\partial \xi_i^4} = 0. \tag{14}$$

The coefficients A, ReB, WeN are given in the appendix.

Using the substitution

$$\begin{aligned} &\xi_i = (\mathrm{We}|N|/\mathrm{Re}|B|)^{1/2}\xi, \quad H_{1i} = H(\mathrm{Re}|B|)^{3/2}/(A(\mathrm{We}|N|)^{1/2}), \\ &\tau_2 = \tau \mathrm{We}|N|/(\mathrm{Re}|B|)^2 \end{aligned} \tag{15}$$

we change Eq. (14) to the form

$$\frac{\partial H}{\partial \tau}+H\frac{\partial H}{\partial \xi}+\operatorname{sgn} B\frac{\partial^2 H}{\partial \xi^2}+\operatorname{sgn} N\frac{\partial^4 H}{\partial \xi^4}=0, \tag{16}$$

i.e., depending on the signs of the coefficients, we actually have four different equations:

$$\frac{\partial H}{\partial \tau}+H\frac{\partial H}{\partial \xi}+\frac{\partial^2 H}{\partial \xi^2}+\frac{\partial^4 H}{\partial \xi^4}=0; \tag{17}$$

$$\frac{\partial H}{\partial \tau}+H\frac{\partial H}{\partial \xi}-\frac{\partial^2 H}{\partial \xi^2}+\frac{\partial^4 H}{\partial \xi^4}=0; \tag{18}$$

$$\frac{\partial H}{\partial \tau}+H\frac{\partial H}{\partial \xi}+\frac{\partial^2 H}{\partial \xi^2}-\frac{\partial^4 H}{\partial \xi^4}=0; \tag{19}$$

$$\frac{\partial H}{\partial \tau}+H\frac{\partial H}{\partial \xi}-\frac{\partial^2 H}{\partial \xi^2}-\frac{\partial^4 H}{\partial \xi^4}=0. \tag{20}$$

Equation (17) is often encountered in the modeling of the nonlinear behavior of disturbances in active media. In particular, for Re ~ 1 this equation is obtained in examining waves on the surface of one freely-flowing liquid film [1]. For such a flow, the signs of the coefficients A, B, and N depend on the specific values of the parameters ν, σ, ρ, and h and on which of the two steady (in the first approximation) waves are being examined. Also, as is clear from (15), in all four cases $H_{1i} \sim -H$ if $A < 0$.

If Eq. (17) is known to have periodic and soliton solutions in the form of steady traveling waves (see [1, 3-6] for example), $H = H(\xi - c\tau)$, then (18) and (19) have no such solutions. In fact, it is easily shown that the following relation is valid for any solution of Eq. (16) which is periodic with respect to ξ (with the wavelength λ)

$$\frac{1}{2}\frac{\partial}{\partial \tau}\int_0^\lambda H^2\,d\xi=\int_0^\lambda\left[\operatorname{sgn} B\left(\frac{\partial H}{\partial \xi}\right)^2-\operatorname{sgn} N\left(\frac{\partial^2 H}{\partial \xi^2}\right)^2\right]d\xi. \tag{21}$$

Since the left side of Eq. (21) is equal to zero for a steady traveling wave, then it is clear that this equation can be satisfied for Eqs. (18) and (19) only if H = 0.

It follows from (21) that any solution of Eq. (18) which is periodic with respect to ξ will decay with time, while the analogous solution for (19) will increase without limit.

Equation (20) is formally equivalent to (16), since it is reduced to it by means of the substitution

$$\tau \to -\tau,\ H \to -H. \tag{22}$$

Thus, (20) also has solutions in the form of steady traveling waves. However, if such periodic solutions of Eq. (17) include solutions which (as was shown in [1, 6], for example) are stable against all infinitesimal plane perturbations and if only one or two of the modes are increasing for many of the unstable solutions, then all of the steady traveling solutions of Eq. (20) will be highly unstable. As is clear from (22), all or nearly all such disturbances will be increasing (all of them will be if the solutions of Eq. (17) for the corresponding wave numbers are stable).

It is clear from the above that initial flow (1) is most stable against small but finite two-dimensional disturbances when it has parameters at which the coefficients A, B, and N for either of the steady traveling solutions of system (9) are such that (14) reduces to Eq. (18) for these coefficients. In this case, disturbances decay over time.

If the description of the evolution of only one of the waves (13) reduces to (18) and if Eq. (17) exists for the other wave, then a fairly well-developed wave pattern will be seen at the interfaces.

If Eq. (14) can be reduced to even one of the waves (13), then the wave pattern will be highly unstable. Thus, the initial flow (1) will also be highly unstable.

The situation will be most unstable if the parameters of the flow (1) are such that we arrive at Eq. (19) for one of the waves (13). In this case, disturbances quickly grow to amplitudes at which the present approximation is rendered invalid.

The above conclusions were not arrived at in rigorous fashion, being based on the analysis only of a certain class of disturbances – those which satisfy Eq. (13). To study the stability of flow (1) relative to all possible two-dimensional disturbances, it will be necessary to take the general solution of system (9) and derive the corresponding system of the second approximation for it from (11). Such an attempt is beyond the scope of the present work.

Despite the limited nature of the approach that has been taken here, the results which follow from the analysis of Eq. (14) can be used at least as estimates.

In the study of the combined flow of films of specific liquids, the coefficients in (14) will be functions only of their relative thickness. Thus, by appropriately selecting the thicknesses of these films, we can control their flow regimes within certain limits.

Thus, for example, calculations show that in the case of a water–benzene system, Eq. (14) reduces to (20) for one of the waves (13) at all values of h from the range [$0.5 \le h \le 5$]. Equation (17) exists for the second wave at $0.5 \lesssim h \lesssim 3.5$, while at $h > 3.5$ Eq. (14) reduces to (18). With allowance for the above stipulations, it can be expected that such a two-layer film will be more stable against disturbances in the second case than in the first case.

A benzene–water film serves as the opposite example. With values of relative thickness $h \ge 2.8$, the evolution of one of the waves is described by Eq. (19), i.e., in this case nonlinear effects do not keep disturbances from growing and such a film will evidently be quickly destroyed.

If the first film is olive oil and the second is water, then Eq. (18) is valid for one of the waves for all h from the investigated range and Eq. (17) is valid for the other wave. Here, $A < 0$ for the last wave on the interval [$0.5 \lesssim h \lesssim 1.1$]. It is not hard to show that unusual wave profiles may exist at the interface between the films.

In fact, as is known from [3, 5, 6], Eq. (17) has families of periodic steady traveling solutions which pass to the limit as the wave number α approaches zero in soliton solutions. For these solitons, $|H_{min}| > |H_{max}|$, so they can be referred to as soliton-depressions [6]. Their leading edge is monotonic, while decaying oscillations exist on the trailing edge. Since $A < 0$ for the investigated thicknesses, then it follows from (14), (17), and (15) that when the wave number α is small enough, it is possible to have periodic regimes in which the profiles have the form of a sequence of steady traveling "positive" soliton-elevations. The latter will have a monotonic leading edge and an oscillating trailing edge. This situation is opposite that seen for a freely-flowing single film, where such profiles are in principle impossible due to the positiveness of all of the coefficients of initial equation (14). In this case, the oscillations for the "positive" solitons will always occur on the leading edge [3, 5, 6].

We thank Academician V. E. Nakoryakov for formulating the problem examined here.

APPENDIX

$$r_1 = a\left[(1-\rho)h_{10} + h_{20}/\nu\right], \quad r_2 = ah_{20}/\nu,$$

$$r_3 = a\left[(\rho(h-2) + 1 - (1-h)/\nu)h_{10} + (\mu - 1)h_{20}/\nu\right],$$

$$r_4 = \frac{1}{2}\left[-\frac{\partial r_3}{\partial x} + \frac{a}{\nu}(1-h)(1-\mu)\frac{\partial h_{10}}{\partial x}\right],$$

$$f_1 = -\frac{\partial h_{10}}{\partial \tau_2} + a\left[\rho(3-h) - 2\right]h_{10}\frac{\partial h_{10}}{\partial x} + B_1\frac{\partial^2 h_{10}}{\partial x^2} + D_1\frac{\partial^2 h_{10}}{\partial x\,\partial \tau_1} -$$

$$-\frac{a\mathrm{We}}{3}\frac{\partial^4 h_{10}}{\partial x^4} - \rho a\frac{\partial}{\partial x}(h_{10}h_{20}) + B_{12}\frac{\partial^2 h_{20}}{\partial x^2} + D_{12}\frac{\partial^2 h_{20}}{\partial x\,\partial \tau} - \frac{a\mathrm{We}\sigma}{6}(3h-1)\frac{\partial^4 h_{20}}{\partial x^4},$$

$$f_2 = -\frac{\partial h_{20}}{\partial \tau_2} + \frac{2a}{\nu}\left[1 - h - \mu\right]h_{20}\frac{\partial h_{20}}{\partial x} + B_2\frac{\partial^2 h_{20}}{\partial x^2} + D_2\frac{\partial^2 h_{20}}{\partial x\,\partial \tau_1} + N_2\frac{\partial^4 h_{20}}{\partial x^4} +$$

$$+\frac{a}{\nu}\left[2(h-1)(\mu-1) + \nu - \mu\right]\frac{\partial}{\partial x}(h_{10}h_{20}) + 2a(\rho-1)h_{10}\frac{\partial h_{10}}{\partial x} - \frac{a\mathrm{We}\sigma}{6}(3h-1)\frac{\partial^4 h_{10}}{\partial x^4} + B_{21}\frac{\partial^2 h_{10}}{\partial x^2} + D_{21}\frac{\partial^2 h_{10}}{\partial x\,\partial \tau},$$

$$B_1 = \mathrm{Re}\, a^2 \left\{ \frac{3(1-\rho)}{40}(1+\rho(h-1)) + \frac{\rho}{12\nu^2}[(\mu(h-2)+1+\nu-h) \times \right.$$

$$\left. \times [3(h-1)(\nu-1+2(h-1)(1-\mu)) + h^3 - 1] + \frac{3}{2}(\mu(2h-3)+\nu+2(1-h))(1-h)^2] \right\},$$

$$D_1 = \mathrm{Re}\, a \left[\frac{5(1-\rho)}{24} + \frac{1-h}{2\mu}(\mu(h-2)+\nu+1-h) \right],$$

$$B_{12} = \mathrm{Re}\, a^2\rho \left\{ \frac{3}{40}(1+\rho(h-1)) + \frac{1}{\nu^2}[(\nu-1+2(h-1)(1-\mu))(h-1) \times \right.$$

$$\left. \times (h-1+2\mu)/8 + (h^3-1)(h+\mu-1)/12 - (\mu-1)(h-1)^2/8] \right\},$$

$$D_{12} = \mathrm{Re}\, a\rho \left[\frac{5}{24} + \frac{(h-1)(h-1+2\mu)}{4\nu} \right],$$

$$N_2 = \frac{a\mathrm{We}\sigma}{6\mu}((1-3\mu)h^2 + (6h^2-9h+2)(1-\mu)),$$

$$B_2 = \mathrm{Re}\, a^2 \left\{ \rho(1+\rho(h-1))\left(\frac{h}{8}-\frac{1}{20}\right) + \frac{1}{2\nu^3}[(\nu-1+2(h-1)(1-\mu)) \times \right.$$

$$\left. \times (S_1+(\mu-1)S_3) + 2hS_2 + (\mu-1)(S_1+(2-h)S_3)] \right\},$$

$$D_2 = \mathrm{Re}\, a \left[\rho\left(\frac{5}{24}-\frac{h-1}{3}\right) + (S_1+S_3(\mu-1))/\nu^2 \right],$$

$$B_{21} = \mathrm{Re}\, a^2 \left\{ (1-\rho)(1+\rho(h-1))\left(\frac{h}{8}-\frac{1}{20}\right) + \frac{1}{2\nu^3}[[(\nu-1+2(h-1) \times \right.$$

$$\left. \times (1-\mu))S_3 + 2S_2](\mu(h-2)+\nu+1-h) - (hS_3-S_1)(\mu(3-2h)-\nu-2(1-h))] \right\},$$

$$D_{21} = \mathrm{Re}\, a \left\{ \left(\frac{5}{24}-\frac{h-1}{3}\right)(1-\rho) + \frac{1}{\nu^2}(\mu(h-2)+\nu+1-h)S_3 \right\},$$

$$S_1 = \frac{1-h^4}{24} + \mu\frac{h^2-1}{4} + \frac{(h^2-1)h^2}{4} + (1-h)\left(\mu\frac{1-h^2}{2} - \frac{1}{6} + \frac{h^2}{2}\right),$$

$$S_2 = \frac{1-h^5}{120} + \mu\frac{h^3-1}{12} + \frac{(h^2-1)h^3}{12} + (1-h)\left(\mu\frac{1-h^2}{2} - \frac{1}{24} + \frac{h^3}{6}\right),$$

$$S_3 = \frac{1-h^3}{6} + \mu\frac{h-1}{2} + \frac{h^2-1}{2}h + (1-h)\left(\mu(1-h) - \frac{1}{2} + h\right),$$

$$P = (2c_i - a_1 - a_2)/(a_2 - c_i),\ P_1 = (c_i - a_1)/(c_i - a_2),$$

$$A = \{\rho(3-h) - 2(a_1-c_i)(\rho-1)/b_2 - 2\rho(c_i-a_1)/b_1 - 2 + 2P_1 \times$$

$$\times (2(h-1)(\mu-1) + \nu - \mu + (c_i-a_1)(1-h-\mu))/\nu\}a/P,$$

$$\mathrm{Re}B = \{B_1 - c_iD_1 + (B_{12}-c_iD_{12})(c_i-a_1)/b_1 + P_1(B_2-c_iD_2) -$$

$$-(a_1-c_i)(B_{21}-c_iD_{21})/b_2\}/P,$$

$$\mathrm{We}N = -\{a\mathrm{We}(2+\sigma(3h-1)(c_i-a_1)(1/b_1+1/b_2))/6 - P_1N_2\}/P.$$

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